Exercise 74

(a) Where does the normal line to the ellipse $x^2 - xy + y^2 = 3$ at the point $(-1, 1)$ intersect the ellipse a second time?

(b) Illustrate part (a) by graphing the ellipse and the normal line.

Solution

Differentiate both sides with respect to x.

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(3)$$

Use the chain rule to differentiate $y = y(x)$.

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(3)$$

$$2x - \left[\frac{d}{dx}(x)\right] y - x\left[\frac{d}{dx}(y)\right] + 2y\frac{dy}{dx} = 0$$

$$2x - (1)y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

Solve for dy/dx.

$$2x - y + (-x + 2y)\frac{dy}{dx} = 0$$

$$(-x + 2y)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{-x + 2y}$$

The slope of the tangent line at $(-1, 1)$ is

$$m = \frac{(1) - 2(-1)}{-(-1) + 2(1)} = 1,$$

so the slope of the normal line is the negative reciprocal.

$$m_\perp = -\frac{-(-1) + 2(1)}{(1) - 2(-1)} = -1$$

Use the point-slope formula to obtain the equation of the normal line.

$$y - 1 = -1(x - (-1))$$

$$y - 1 = -(x + 1)$$

$$y - 1 = -x - 1$$

$$y = -x$$

In order to find where the normal line intersects the ellipse, solve the following system of equations.

$$\left.\begin{aligned} y &= -x \\ x^2 - xy + y^2 &= 3 \end{aligned}\right\}$$

Substitute the formula for y into the second equation and solve for x.

$$x^2 - x(-x) + (-x)^2 = 3$$

$$x^2 + x^2 + x^2 = 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

The y-value corresponding to $x = 1$ is $y = -(1) = -1$, so the second point where the normal line intersects the ellipse is $(1, -1)$.